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Eigenvalue estimates for the Dirac operator depending on the Weyl tensor ☆

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Abstract

We prove new lower bounds for the first eigenvalue of the Dirac operator on compact manifolds whose Weyl tensor or curvature tensor, respectively, is divergence-free. In the special case of Einstein manifolds, we obtain estimates depending on the Weyl tensor. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

If M^n is a compact Riemannian spin manifold with positive scalar curvature R, then each eigenvalue λ of the Dirac operator D satisfies the inequality

$$\lambda^2 \geq \frac{nR_0}{4(n-1)},$$

where R_0 is the minimum of R on M^n. The estimate is sharp in the sense that there exist manifolds for which the lower bound is an eigenvalue λ_1 of D. In this case, M^n must be an Einstein space (see [2]). A generalization of this inequality was proved in this paper [6], in

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which a conformal lower bound for the spectrum of the Dirac operator appeared. Moreover, for special Riemannian manifolds, better estimates for the eigenvalues of the Dirac operator are known, see [7,8]. In this paper [5], we proved an estimate for the eigenvalues of the Dirac operator depending on the Ricci tensor in case that the curvature tensor is harmonic. In this note we will prove an estimate of the Dirac spectrum depending on the scalar curvature and on the Weyl tensor for manifolds with divergence-free Weyl tensor. In particular, we prove that a compact, conformally Ricci-flat manifold with certain nontrivial conformal invariant ν_0 does not admit any harmonic spinors. A second application of our result refers to symmetric spaces of compact type. In this case, our inequality may be simplified and depends mainly on the scalar curvature and on the length of the Weyl tensor. Under the assumption that the curvature tensor is harmonic, we prove in Section 4 an estimate depending on the Ricci tensor as well as on the Weyl tensor.

2. Curvature endomorphisms of the spinor bundle

Let M^n be a Riemannian spin manifold of dimension $n \geq 4$ with Riemannian metric g and spinor bundle S. By ∇ we denote the covariant derivative induced by g on the tangent bundle TM^n as well as the corresponding derivative in the spinor bundle S. For any vector field X, Y on M^n, we use the notation

$$\nabla_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$$

for the tensorial derivatives of second order. By K and C we denote the Riemannian curvature tensor and the curvature tensor of S, respectively. Then, for any vector field X, Y, Z and any spinor field ψ, we have

$$K(X,Y)(Z) = \nabla_{X,Y} Z - \nabla_{Y,X} Z, \qquad C(X,Y)\psi = \nabla_{X,Y}\psi - \nabla_{Y,X}\psi.$$

Given a local frame of vector fields $(X_1, \ldots, X_n)$, we denote by $(X^1, \ldots, X^n)$ the associated frame defined by $X^k := g^{kl} X_l$, where (g^{kl}) is the inverse of the matrix (g_{kl}) with $g_{kl} = g(X_k, X_l)$. For the reader's convenience, we summarize some well-known identities:

$$C(X,Y) = \tfrac{1}{4} X_k \cdot K(X,Y)(X^k) = -\tfrac{1}{4} K(X,Y)(X^k) \cdot X_k, \tag{1}$$

$$C(X,Y) \cdot Z - Z \cdot C(X,Y) = K(X,Y)(Z), \tag{2}$$

$$X_k \cdot C(X^k, Y) = \tfrac{1}{2}\mathrm{Ric}(Y) = C(X^k, Y) \cdot X_k, \tag{3}$$

$$X_k \cdot K(X, X^k)(Y) = 2C(X,Y) + g(\mathrm{Ric}(X), Y), \tag{4}$$

$$X^k \cdot \nabla_{X,X_k}\psi = \nabla_X D\psi, \tag{5}$$

$$X^k \cdot \nabla_{X_k,X}\psi = \nabla_X D\psi + \tfrac{1}{2}\mathrm{Ric}(X) \cdot \psi, \tag{6}$$

$$X^k \cdot \mathrm{Ric}(X_k) = \mathrm{Ric}(X_k) \cdot X^k = -R, \tag{7}$$

where Ric is the Ricci tensor, R the scalar curvature and D the Dirac operator locally defined by $\mathrm{Ric}(X) := K(X, X_k)(X^k)$, $R = g(\mathrm{Ric}(X_k), X^k)$ and $D\psi = X^k \cdot \nabla_{X_k}\psi$, respectively.

For any vector field X and Y, we consider the endomorphism $E(X, Y)$ of S or $\Gamma(S)$, respectively, locally given by

$$E(X, Y) := -C(X_k, Y) \cdot C(X^k, X).$$

Since $C(X, Y)$ is anti-selfadjoint with respect to the Hermitian scalar product $\langle\cdot,\cdot\rangle$ on S, the endomorphism $E(X, Y)$ has a similar property

$$C(X, Y)^* = -C(X, Y), \qquad E(X, Y)^* = E(Y, X). \tag{8}$$

The endomorphism F of S defined as the contraction of E,

$$F := E(X_k, X^k) = -C(X_k, X_l)C(X^k, X^l),$$

is selfadjoint and nonnegative, i.e.,

$$F^* = F, \quad \langle F\psi, \psi\rangle \geq 0. \tag{9}$$

We denote by W the Weyl tensor of the Riemannian manifold and we introduce the following endomorphisms acting in the spinor bundle:

$$B(X, Y) := \tfrac{1}{4} X_k \cdot W(X, Y)(X^k), \qquad G(X, Y) := -B(X_k, Y)B(X^k, X),$$
$$H := G(X_k, X^k) = -B(X_k, X_l)B(X^k, X^l).$$

We collect some properties of these endomorphisms:

$$B(X, Y)^* = -B(X, Y), \qquad G(X, Y)^* = G(Y, X), \tag{10}$$

$$X_k \cdot B(X^k, Y) = 0 = B(X^k, Y) \cdot X_k, \tag{11}$$

$$H^* = H, \qquad \langle H\psi, \psi\rangle \geq 0, \tag{12}$$

$$B(X, Y) \cdot Z - Z \cdot B(X, Y) = W(X, Y)(Z), \tag{13}$$

$$F = H + \frac{1}{2(n-2)}\left|\mathrm{Ric} - \frac{R}{n}\right|^2 + \frac{R^2}{4n(n-1)}. \tag{14}$$

Let $\nu(x)$ denote the smallest eigenvalue of H at the point $x \in M^n$. Its infimum

$$\nu_0 := \inf\{\nu(x) : x \in M^n\} \geq 0$$

will occur in our estimates of the eigenvalues of the Dirac operator D. If ψ is a parallel spinor ($\nabla\psi = 0$), then, for all vector fields X and Y, it follows that $C(X, Y) \cdot \psi = 0$ and, hence, $F\psi = 0$. Thus, the relation (14) shows that the Ricci tensor as well as the function $\nu \geq 0$ are obstructions for the existence of parallel spinors. Moreover, the Schrödinger–Lichnerowicz formula

$$\nabla^*\nabla = D^2 - \tfrac{1}{4}R \tag{15}$$

implies that, on compact manifolds with vanishing scalar curvature, each harmonic spinor is parallel. Hence, as M^n is compact, Ricci flat and $\nu_0 > 0$, there are no harmonic spinors.

3. Estimate for manifolds with divergence-free Weyl tensor

In this section, we assume that the Weyl tensor W of M^n is divergence-free, i.e., for all vector fields Y and all local frames $(X_1, \dots, X_n)$, W satisfies the condition

$$(\nabla_{X_k} W)(X^k, Y) = 0. \tag{16}$$

By definition of B, this implies

$$(\nabla_{X_k} B)(X^k, Y) = 0. \tag{17}$$

For any real parameter $t \in \mathbb{R}$, we consider the differential operator $P^t : \Gamma(S) \to \Gamma(TM^n \otimes S)$ locally defined by $P^t\psi := X^k \otimes P^t_{X_k}\psi$ and

$$P^t_X\psi := \nabla_X\psi + \frac{1}{n} X \cdot D\psi - tB(X, X^k) \cdot \nabla_{X_k}\psi. \tag{18}$$

Using the twistor operator $\mathcal{D} : \Gamma(S) \to \Gamma(TM^n \otimes S)$ given by $\mathcal{D}\psi := X^k \otimes (\nabla_{X_k}\psi + (1/n)X_k \cdot D\psi)$, this may be rewritten as

$$P^t\psi := \mathcal{D}\psi - tX^k \otimes B(X_k, X^l) \cdot \nabla_{X_l}\psi. \tag{19}$$

The image of $\mathcal{D}$ is contained in the kernel of the Clifford multiplication, i.e., it holds that

$$X^k \cdot \mathcal{D}_{X_k}\psi = 0. \tag{20}$$

Thus, by (11) and (20), it follows that:

$$X^k \cdot P^t_{X_k}\psi = 0. \tag{21}$$

Lemma 3.1. *Suppose that the Weyl tensor W is divergence-free. Then, any spinor field ψ satisfies the equation*

$$|P^t\psi|^2 = |\nabla\psi|^2 - \frac{1}{n}|D\psi|^2 - t\langle H\psi, \psi\rangle + t^2\langle G(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle - 2t\,\mathrm{div}\langle B\psi, \nabla\psi\rangle, \tag{22}$$

where $\langle B\psi, \nabla\psi\rangle$ is the vector field locally defined by $\langle B\psi, \nabla\psi\rangle := \langle B(X^i, X^k)\psi, \nabla_{X_k}\psi\rangle X_i$.

Proof. Using the formulas (10), (11) and (18), we calculate

$$|P^t\psi|^2 = \langle P^t_{X_i}\psi, P^t_{X^i}\psi\rangle = |\mathcal{D}\psi|^2 - 2t\langle\nabla_{X_i}\psi, B(X^i, X^k)\nabla_{X_k}\psi\rangle + t^2\langle B(X_i, X^k)\nabla_{X_k}\psi, B(X^i, X^l)\nabla_{X_l}\psi\rangle.$$

Thus, we obtain

$$|P^t\psi|^2 = |\nabla\psi|^2 - \frac{1}{n}|D\psi|^2 - 2t\langle\nabla_{X_i}\psi, B(X^i, X^k)\nabla_{X_k}\psi\rangle + t^2\langle G(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle. \tag{$*$}$$

Let $x \in M^n$ be any point and let $(X_1, \dots, X_n)$ be an orthonormal frame in a neighborhood of x with $(\nabla X_k)_x = 0$. Then, we have at the point x

$$\begin{aligned}\langle \nabla_{X_i}\psi, B(X^i, X^k)\nabla_{X_k}\psi\rangle &= X_i(\langle \psi, B(X^i, X^k)\nabla_{X_k}\psi\rangle) \\ &\quad - \langle \psi, (\nabla_{X_i}B)(X^i, X^k)\nabla_{X_k}\psi\rangle \\ &\quad - \langle \psi, B(X^i, X^k)\nabla_{X_i}\nabla_{X_k}\psi\rangle \\ &\overset{(17)}{=} \operatorname{div}\langle B\psi, \nabla\psi\rangle - \tfrac{1}{2}\langle \psi, B(X^i, X^k)C(X_i, X_k)\psi\rangle \\ &\overset{(11)}{=} \operatorname{div}\langle B\psi, \nabla\psi\rangle - \tfrac{1}{2}\langle \psi, B(X^i, X^k)B(X_i, X_k)\psi\rangle \\ &= \operatorname{div}\langle B\psi, \nabla\psi\rangle + \tfrac{1}{2}\langle \psi, H\psi\rangle.\end{aligned}$$

Inserting this into (∗) we obtain (22). □

Let us introduce the number μ_0 measuring the maximum of the norm of the Weyl tensor,

$$\mu_0^2 = \max\left(\tfrac{1}{16}\|W_{XYij}X^i \cdot X^j\|^2 : \langle X, Y\rangle = 0, |X| = |Y| = 1\right).$$

Then, for any point $x \in M^n$ and any orthonormal basis $(X_1, \dots, X_n)$ of $T_x M^n$, we have

$$\begin{aligned}|\langle G(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle| &= |\langle B(X_i, X^k)\nabla_{X_k}\psi, B(X^i, X^l)\nabla_{X_l}\psi\rangle| \\ &\le \sum_{i,k,l} |\langle B(X_i, X_k)\nabla_{X_k}\psi, B(X_i, X_l)\nabla_{X_l}\psi\rangle| \\ &\le \sum_{i,k,l} |B(X_i, X_k)\nabla_{X_k}\psi| \cdot |B(X_i, X_l)\nabla_{X_l}\psi| \\ &\le \sum_{i,k,l} \|B(X_i, X_k)\|\|B(X_i, X_l)\||\nabla_{X_k}\psi||\nabla_{X_l}\psi| \\ &\le n\mu_0^2 \left(\sum_{k,l} |\nabla_{X_k}\psi|\,|\nabla_{X_l}\psi|\right) = n\mu_0^2\left(\sum_k |\nabla_{X_k}\psi|\right)^2 \\ &\le n^2\mu_0^2\left(\sum_k |\nabla_{X_k}\psi|^2\right) = n^2\mu_0^2|\nabla\psi|^2.\end{aligned}$$

Thus, we obtain the estimate

$$|\langle G(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle| \le n^2\mu_0^2|\nabla\psi|^2. \tag{23}$$

Furthermore, let us denote the minimum of R on M^n by R_0. Then, applying (15), (22) and (23) to an eigenspinor ψ of the Dirac operator ($D\psi = \lambda\psi$), we have

$$\begin{aligned}0 &\le \int_{M^n} |P^t\psi|^2 \\ &= \int_{M^n}\left(\frac{n-1}{n}\lambda^2|\psi|^2 - \frac{R}{4}|\psi|^2 - t\langle H\psi, \psi\rangle + t^2\langle G(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle\right) \\ &\le \left(\frac{n-1}{n}\lambda^2 - \frac{R_0}{4} - \nu_0 t + n^2\mu_0^2\left(\lambda^2 - \frac{R_0}{4}\right)t^2\right) \cdot \int_{M^n}|\psi|^2.\end{aligned}$$

This implies the inequality

$$\frac{n-1}{n}\lambda^2 - \frac{R_0}{4} - \nu_0 t + n^2\mu_0^2\left(\lambda^2 - \frac{R_0}{4}\right)t^2 \geq 0,$$

which is equivalent to

$$\lambda^2 \geq \frac{n}{4}\cdot\frac{R_0 + 4\nu_0 t + n^2 R_0\mu_0^2 t^2}{(n-1) + n^3\mu_0^2 t^2} \quad (\forall t \in \mathbb{R}). \tag{24}$$

By computing the maximum of the right side with respect to the parameter t, we obtain the main theorem of this section.

Theorem 3.1. *Let M^n be a compact Riemannian spin manifold with divergence-free Weyl tensor. Then, for any eigenvalue λ of the Dirac operator, we have the inequality*

$$\lambda^2 \geq \frac{nR_0}{4(n-1)} + \frac{2\nu_0^2}{n\mu_0^2(R_0 + \sqrt{R_0^2 + (n-1/n)(4\nu_0/\mu_0)^2})}. \tag{25}$$

Moreover, for $R_0 < 0$, this lower bound is positive if the condition $\nu_0 > \frac{1}{2}n|R_0|\mu_0$ is satisfied.

The divergence of the Weyl tensor is given by the well-known identity

$$(\nabla_{X_k} W)(X, Y)(X^k) = (n-3)((\nabla_X T)(Y) - (\nabla_Y T)(X)), \tag{26}$$

where the tensor T is defined by

$$T(X) := \frac{1}{n-2}\left(\frac{R}{2(n-1)}X - \mathrm{Ric}(X)\right).$$

In particular, any Einstein manifold has a divergence-free Weyl tensor.

Corollary 3.1. *The inequality* (25) *holds for any compact Einstein spin manifold*.

Corollary 3.2. *Let M^n be a compact Riemannian spin manifold with divergence-free Weyl tensor and vanishing scalar curvature. Then, we have the estimate*

$$\lambda^2 \geq \frac{\nu_0}{2\mu_0\sqrt{n(n-1)}}. \tag{27}$$

Consider a compact Riemannian spin manifold (M^n, g^*) and suppose that there exists a conformally equivalent Ricci-flat metric $g = e^f \cdot g^*$. Since the Weyl tensor is a conformal invariant, the condition $\nu_0 > 0$ is conformally invariant, too. The same is true for the dimension of the space of harmonic spinors.

Corollary 3.3. *A compact, conformally Ricci-flat spin manifold with $\nu_0 > 0$ does not admit harmonic spinors*.

In the case of an even-dimensional manifold, the spinor bundle splits into the bundle of positive and negative spinors, respectively. We can introduce two smallest eigenvalues $\nu_0^{\pm}$ and Corollary 3.3 holds in any of these two bundles.

The lower bound for the eigenvalues of the Dirac operator proved in Theorem 3.1 depends on the minimum of the scalar curvature, on the maximum μ_0^2 and on the smallest eigenvalue of the nonnegative endomorphism

$$H := -\frac{1}{16}\sum_{k,l}\sum_{\alpha,\beta,\gamma,\delta} W_{kl\alpha\beta}W_{kl\gamma\delta}X^{\alpha}\cdot X^{\beta}\cdot X^{\gamma}\cdot X^{\delta}$$

acting on the spinor bundle. Using the grading of the Clifford algebra, we decompose the endomorphism $H := H_0 + H_2 + H_4$ into three parts, where H_0 is a scalar and H_2, H_4 are the elements of the Clifford algebra of degree 2 and 4, respectively. It is easy to compute H_0,

$$H_0 = \tfrac{1}{8}|W|^2.$$

Since H and H_4 are hermitean and H_2 is anti-hermitean, we conclude that $H_2 = 0$. Consequently, we obtain the formula

$$\nu_0 = \min(\tfrac{1}{8}|W|^2 + \langle H_4\cdot\psi,\psi\rangle : |\psi| = 1),$$

where H_4 is given by

$$H_4 = -\frac{1}{2}\sum_{k,l}\sum_{\alpha<\beta<\gamma<\delta}(W_{kl\alpha\beta}W_{kl\gamma\delta} - W_{kl\alpha\gamma}W_{kl\beta\delta} + W_{kl\alpha\delta}W_{kl\beta\gamma})X^{\alpha}\cdot X^{\beta}\cdot X^{\gamma}\cdot X^{\delta}.$$

In the four-dimensional case, the endomorphism $X^1\cdot X^2\cdot X^3\cdot X^4$ acts on the two parts of the spinor bundle $S = S^+\oplus S^-$ by multiplication by ± 1 and we obtain the simple formula

$$\nu_0 = \min(\tfrac{1}{16}|W + *W|^2, \tfrac{1}{16}|W - *W|^2).$$

For example, consider the square of the Dirac operator acting on the bundle S^+ of positive spinors over a four-dimensional Kähler–Einstein manifold. The positive part W^+ of the Weyl tensor acting on Λ^2_+ has the diagonal form $W^+ = \mathrm{diag}(-\frac{1}{6}R, -\frac{1}{6}R, \frac{1}{3}R)$ (see [1]) and, consequently, we compute

$$\nu_0^+ = \frac{1}{6}R^2, \qquad \mu_0^2 = \frac{1}{8\cdot 9}R^2.$$

The estimate of Theorem 3.1 yields the inequality $\lambda^2 \geq \frac{1}{2}R$ and this is precisely the lower bound for the eigenvalues of the Dirac operator on any four-dimensional Kähler manifold (see [4,7]).

On the other hand, if the Weyl tensor of the manifold M^n ($n \geq 5$) satisfies the relation

$$\sum_{k,l}(W_{kl\alpha\beta}W_{kl\gamma\delta} + W_{kl\alpha\gamma}W_{kl\delta\beta} + W_{kl\alpha\delta}W_{kl\beta\gamma}) = 0,$$

then H_4 vanishes. This situation occurs, for example, if M^n is an irreducible symmetric space of compact type. The proof is an easy computation using the well-known formulas for the curvature tensor of a symmetric space, which is why we shall only sketch it shortly. First, we remark that the following relation holds on any Einstein space:

$$\sum_{k,l} W_{kl\alpha\beta} W_{kl\gamma\delta} = \sum_{k,l} R_{kl\alpha\beta} R_{kl\gamma\delta} + \frac{4R}{n(n-1)} R_{\alpha\beta\gamma\delta} + \frac{2R^2}{n^2(n-1)^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}).$$

In case of a symmetric space $M^n = G/K$, we obtain the formula

$$\sum_{k,l} R_{kl\alpha\beta} R_{kl\gamma\delta} = (\Omega^G - \Omega^K) R_{\alpha\beta\gamma\delta},$$

where Ω^G and Ω^K are the Casimir operators acting on the Lie algebras of the group G and K, respectively. The result follows now from the first Bianchi identity of the curvature tensor.

Proposition 3.1. *Let M^n be an irreducible symmetric space of compact type. Then,*

$$\nu_0 = \tfrac{1}{8}|W|^2.$$

It is well known that a compact, symmetric space with $\lambda^2 = (nR/4(n-1)) > 0$ is a sphere. This result follows immediately from our inequality. Indeed, in this case M^n is an irreducible Einstein manifold (see [2,3]) and, furthermore, we have $\nu_0 = 0$. Since the manifold is symmetric, we conclude that the Weyl tensor vanishes, i.e., M^n is a space of constant curvature.

4. Estimates in case of divergence-free curvature tensor

In this section, we assume that the Riemannian curvature tensor K is divergence-free, i.e., locally we have the equality

$$(\nabla_{X_k} K)(X^k, Y) = 0 \tag{28}$$

for each vector field Y. The Bianchi identity implies the general relation

$$(\nabla_{X_k} K)(X, Y)(X^k) = (\nabla_Y \mathrm{Ric})(X) - (\nabla_X \mathrm{Ric})(Y). \tag{29}$$

Thus, (28) is equivalent to

$$(\nabla_X \mathrm{Ric})(Y) = (\nabla_Y \mathrm{Ric})(X). \tag{30}$$

In particular, the scalar curvature R is constant. Moreover, (28) implies

$$(\nabla_{X_k} C)(X^k, Y) = 0. \tag{31}$$

For $t \in \mathbb{R}$, we now consider the operator

$$Q^t : \Gamma(S) \to \Gamma(TM^n \otimes S)$$

defined by $Q^t\psi := X^k \otimes Q^t_{X_k}\psi$, and

$$Q^t_X\psi := \mathcal{D}_X\psi - t\cdot C(X, X^k)\cdot\nabla_{X_k}\psi.$$

A straightforward calculation yields

$$|Q^t\psi|^2 = |\mathcal{D}\psi|^2 + 2t\langle C(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle + \frac{t}{n}\mathrm{Re}\langle D\psi, \mathrm{Ric}(X^k)\nabla_{X_k}\psi\rangle + t^2\langle E(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle. \quad (32)$$

Furthermore, by Lemmas 1.2 and 1.4 in [5] and our assumption, there are equations

$$\langle C(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle = \mathrm{div}\langle C\psi, \nabla\psi\rangle - \tfrac{1}{2}\langle\psi, F\psi\rangle, \quad (33)$$

$$\mathrm{Re}\langle\psi, \mathrm{Ric}(X^k)\nabla_{X_k}D\psi\rangle = |\nabla D\psi|^2 - |(D^2 - \tfrac{1}{4}R)\psi|^2 - \tfrac{1}{4}R|\nabla\psi|^2 + \tfrac{1}{4}|\mathrm{Ric}|^2|\psi|^2 + \langle\nabla_{\mathrm{Ric}(X_k)}\psi, \nabla_{X^k}\psi\rangle - \mathrm{div}(X_\psi), \quad (34)$$

where X_ψ is a vector field depending on ψ. From (14) and (32)–(34), we obtain the basic Weitzenböck formula of this section.

Lemma 4.1. *Let M^n be a Riemannian spin manifold with divergence-free curvature tensor and let λ be any eigenvalue of the Dirac operator. Then, for any corresponding eigenspinor ψ and for all $t \in \mathbb{R}$, we have the equation*

$$\begin{aligned}|Q^t\psi|^2 &= |\nabla\psi|^2 - \frac{\lambda^2}{n}|\psi|^2 + \frac{t}{n}\langle\nabla_{\mathrm{Ric}(X_k)}\psi, \nabla_{X^k}\psi\rangle \\ &\quad + \frac{t}{n}\left(\lambda^2 - \frac{R}{4}\right)\left(|\nabla\psi|^2 - \left(\lambda^2 - \frac{R}{4}\right)|\psi|^2\right) - t\langle\psi, H\psi\rangle \\ &\quad - \frac{t}{4n}\left(\frac{n+2}{n-2}\left|\mathrm{Ric} - \frac{R}{n}\right|^2 + \frac{R^2}{n(n-1)}\right)|\psi|^2 + \mathrm{div}\left(2t\langle C\psi, \nabla\psi\rangle - \frac{t}{n}X_\psi\right) \\ &\quad + t^2\langle E(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle.\end{aligned}$$

We consider the curvature tensor K as an endomorphism of $\Lambda^2 M^n$ by the usual definition

$$K(u^i \wedge u^j) := \tfrac{1}{2}g(K(X^i, X^j)(X_k), X_l)u^k \wedge u^l,$$

where $(u^1, \ldots, u^n)$ denotes the coframe dual to the local frame $(X_1, \ldots, X_n)$. Then K is selfadjoint with respect to the scalar product on $\Lambda^2 M^n$ induced by the Riemannian metric g. Let M^n be compact and denote by σ the maximum of the absolute values of the eigenvalues of K on $\Lambda^2 M^n$. Then we estimate the operator norm of the endomorphism $C(X_i, X_j)$ acting on the spinor bundle

$$\begin{aligned}\|C(X_i, X_j)\| &\le \frac{1}{2}\sum_{k<l}|g(K(X_i, X_j)(X_k), X_l)|\cdot\|X_k\cdot X_l\| \\ &= \frac{1}{2}\sum_{k<l}|g(K(X_i, X_j)(X_k), X_l)| \le \frac{1}{2}\binom{n}{2}\sigma.\end{aligned}$$

Using this upper bound, we obtain

$$
\begin{aligned}
|\langle E(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle| &\leq \sum_{i,k,l}\|C(X_i, X_k)\| \cdot \|C(X_i, X_l)\| \cdot |\nabla_{X_k}\psi| \cdot |\nabla_{X_l}\psi| \\
&\leq \frac{n}{4}\binom{n}{2}^2 \sigma^2 \left(\sum_{k,l}|\nabla_{X_k}\psi||\nabla_{X_l}\psi|\right) \\
&\leq \frac{n^2}{4}\binom{n}{2}^2 \sigma^2 \left(\sum_{k}|\nabla_{X_k}\psi|^2\right) = \left(\frac{n}{2}\binom{n}{2}\sigma\right)^2 |\nabla\psi|^2,
\end{aligned}
$$

and, consequently, the inequality

$$
|\langle E(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle| \leq \left(\frac{n}{2}\binom{n}{2}\sigma\right)^2 |\nabla\psi|^2. \tag{35}
$$

If κ denotes the maximum of all eigenvalues of the Ricci tensor on M^n, then we have

$$
\langle \nabla_{\mathrm{Ric}(X_k)}\psi, \nabla_{X^k}\psi\rangle \leq \kappa|\nabla\psi|^2. \tag{36}
$$

Then, by Lemma 4.1, (35) and (36), for all $t \geq 0$, we obtain

$$
\begin{aligned}
\left(\frac{n-1}{n} + \frac{\kappa}{n}t + \left(\frac{n}{2}\binom{n}{2}\sigma\right)^2 t^2\right)\lambda^2 \geq{}& + \frac{1}{4n}\left(\frac{n+2}{n-2}\left|\mathrm{Ric} - \frac{R}{n}\right|_0^2 + \frac{R^2}{n(n-1)}\right)t \\
&+ \nu_0 t + \frac{R}{4}\left(1 + \frac{\kappa}{n}t + \left(\frac{n}{2}\binom{n}{2}\sigma\right)^2 t^2\right),
\end{aligned} \tag{37}
$$

where $|\mathrm{Ric} - (R/n)|_0$ denotes the minimum of the length on M^n. Inserting $\lambda = 0$ in this inequality, we obtain the following theorem.

Theorem 4.1. *Let M^n be a compact Riemannian spin manifold with divergence-free curvature tensor and scalar curvature $R \leq 0$, such that the condition*

$$
\frac{n+2}{n-2}\left|\mathrm{Ric} - \frac{R}{n}\right|_0^2 + \frac{R^2}{n(n-1)} + 4n\nu_0 > |R|\left(\kappa + n^2\binom{n}{2}\sigma\right) \tag{38}
$$

is satisfied. Then, there are no harmonic spinors.

For simplicity, let us introduce the notations

$$
a := 4n(n-1)\nu_0 + \frac{(n-1)(n+2)}{n-2}\left|\mathrm{Ric} - \frac{R}{n}\right|_0^2 - R\left(\kappa - \frac{R}{n}\right),
$$

$$
b := \frac{1}{2}n^2\binom{n}{2}^3\sigma^2, \qquad A_\pm := \sqrt{b^2R^2 + ab(a + R\kappa)} \pm bR.
$$

Moreover, using the new parameter $s := (t/n - 1) \geq 0$, the inequality (37) can be written as

$$\lambda^2 \geq \frac{nR}{4(n-1)} + \frac{s}{4(n-1)} \cdot \frac{a - bRs}{1 + \kappa s + bs^2}. \tag{39}$$

Calculating the maximum of the right side with respect to $s \geq 0$, we obtain our main result.

Theorem 4.2. *Let M^n be a nonflat compact Riemannian spin manifold with divergence-free curvature tensor and let λ be any eigenvalue of the Dirac operator. Then, we have the estimate*

$$\lambda^2 > \frac{nR}{4(n-1)} + \frac{a}{4(n-1)} \cdot \frac{A_-}{2ab + \kappa A_+}. \tag{40}$$

Remark that in case $R > 0, a > 0$, this lower bound is greater than $nR/4(n-1)$. If $R \leq 0$, the lower bound is positive under the condition

$$a + R\kappa > (n-1)|R| \left(\kappa + n^2 \binom{n}{2} \sigma \right). \tag{41}$$

Proof. It remains to show that, for the first eigenvalue λ_1 of D, in (40) equality cannot occur. Let us assume the counterpart. Then, any eigenspinor ψ corresponding to λ_1 satisfies the equation $Q^{t_0}\psi = 0$ with the optimal parameter $t_0 > 0$. By (3) and (20), $Q^{t_0}\psi = 0$ implies

$$\mathrm{Ric}(X^k)\nabla_{X_k}\psi = 0. \tag{$**$}$$

Moreover, the limiting case of (40) implies that in the inequality for $\|C(X_i, X_j)\|$ we have an equality, i.e.,

$$\|C(X_i, X_j)\| = \frac{1}{2}\binom{n}{2}\sigma.$$

Hence, M^n is a space of constant-sectional curvature. In particular, M^n is Einstein ($\mathrm{Ric} = R/n$) and ($**$) implies $0 = R \cdot D\psi = R \cdot \lambda_1 \cdot \psi$. Consequently, the Ricci tensor vanishes and M^n is flat, a contradiction. □

The curvature tensor of any Einstein manifold of dimension $n \geq 4$ is divergence-free. In this special case, we have $\kappa = R/n$ and the number a simplifies to $a = 4n(n-1)\nu_0$.

Example 4.1. An application of the estimate (40) in case of the 3-dimensional manifold $M^3 = S^1 \times S^2$ yields the inequality

$$\lambda^2 > \tfrac{3}{4}(1 + 0.006).$$

Finally, we remark that we generalized Corollary 3.2.

Corollary 4.1. *Let M^n be a compact Riemannian spin manifold with divergence-free curvature tensor and vanishing scalar curvature such that at least one of the numbers*

ν_0 or $|\mathrm{Ric}|_0$, respectively, is not zero. Then, all eigenvalues λ of the Dirac operator satisfy the inequality

$$\lambda^2 > \frac{(n+2)|\mathrm{Ric}|_0^2 + 4n(n-2)\nu_0}{4(n-2)(\kappa + \binom{n}{2}\sigma\sqrt{n(n-1)})}.$$

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